

# On the wave function renormalization for Wilson actions and their 1PI actions

Y. Igarashi<sup>1</sup>, K. Itoh<sup>1</sup>, and H. Sonoda<sup>2</sup>

<sup>1</sup>*Faculty of Education, Niigata University, Niigata 950-2181, Japan*

<sup>2</sup>*Physics Department, Kobe University, Kobe 657-8501, Japan*

.....  
We clarify the relation between the wave function renormalization for Wilson actions and that for the 1PI actions in the exact renormalization group formalism. Our study depends crucially on the use of two independent cutoff functions for the Wilson actions. We relate our results to those obtained previously by Bervillier, Rosten, and Osborn & Twigg.

# 1 Introduction

The purpose of this paper is to explain, as clearly as possible, wave function renormalization for the Wilson actions and their 1PI (one-particle-irreducible) actions. Especially, we wish to elucidate how to incorporate an anomalous dimension of the elementary field into the Wilson and 1PI actions using the formulation of the exact renormalization group (ERG). We are motivated by the unsatisfactory status of the existing literature: we cannot find any simple but general discussion that illuminates wave function renormalization in the ERG formalism. We wish to expose the simple relation between the ERG differential equations for the Wilson actions and those for the corresponding 1PI actions with the anomalous dimension taken into account.

Let us recall that a Wilson action comes with a momentum cutoff, say  $\Lambda$ , and that its dependence on  $\Lambda$  is described by the ERG differential equation. A 1PI action, obtained from the Wilson action by a Legendre transformation, satisfies its own ERG differential equation which is equivalent to that for the Wilson action. There are many reviews available on the subject of ERG[1–9]; we will review everything necessary in Sect. 2 to make the paper self-contained.

An anomalous dimension  $\gamma_\Lambda$  of the elementary field was first introduced to the 1PI actions rather than the Wilson actions by T. Morris in [10, 11]. (The suffix of  $\gamma_\Lambda$  is for potential  $\Lambda$  dependence.) His derivation is rather sketchy and somewhat intuitive, but it has all the correct ingredients. In [12] and [5],  $\gamma_\Lambda$  was introduced differently to the 1PI actions, but the result is trivially related to Morris’s by a rescaling of the field. It is Morris’s result, given as (49) in our paper, which has been extensively used as the general form of the ERG differential equation for the 1PI actions.

Now, what remains confusing in the literature is how to introduce  $\gamma_\Lambda$  to the Wilson actions. Rather recently, in [13] and [14], the starting point was taken as the ERG differential equation given earlier by Ball et al. [15] for Wilson actions. It has  $\gamma_\Lambda$  as the coefficient of a simple linear field transformation. But this simplicity is only apparent; the Legendre transformation, that gives 1PI actions obeying Morris’s differential equation (49), turns out to be very complicated. Earlier in [16] (and again later in [17]) C. Bervillier had introduced  $\gamma_\Lambda$  to the Wilson actions in a way that was later shown in [18] to admit a simple Legendre transformation, leading to (49). But the relation between the results of [13, 14] and those of [16–18] remains to be clarified.

We will base our discussion on a recent work by one of us (H.S.) [19] in which ERG differential equations are formulated in terms of two separate cutoff functions, say  $K_\Lambda$  and  $k_\Lambda$ . (This will be reviewed in Sect. 2.)  $K_\Lambda$  determines the linear part of the ERG differential

equation;  $k_\Lambda$ , together with  $K_\Lambda$ , determines the non-linear part. The anomalous dimension is introduced via the cutoff dependence of  $K_\Lambda$ . Hence, not only the linear part but also the non-linear part of the ERG differential equation depend on  $\gamma_\Lambda$ . It is this equation, given by (36), that we will discuss as the counterpart of the general form (49) for the 1PI actions. A simple and well understood Legendre transformation (32) relates the Wilson and 1PI actions.

With this understanding, the results of [16–18] and those of [13] and [14] become transparent. We will show that the ERG differential equations discussed in [16–18] and in [13, 14] are obtained from the general form (36) by specific choices of our two cutoff functions. We relegate the technical details to appendices. In Appendix B, we will show that the ERG differential equation of [16] is obtained if the cutoff function  $k_\Lambda$  is fixed in terms of  $K_\Lambda$ . Similarly, in Appendix C, we will show that the ERG differential equation of Ball et al. [15] is obtained if we choose  $k_\Lambda$  specifically to remove  $\gamma_\Lambda$  from the non-linear term. (In fact the  $\gamma_\Lambda$  dependence remains hidden in  $k_\Lambda$ .) Consequently we are led to the result of [13, 14] for the Legendre transformation. In passing we note that Wilson’s original ERG differential equation and its variation by Polchinski, both with an anomalous dimension, follow from the general form (36) by appropriate choices of cutoff functions. This will be explained in the second half of Appendix A.

The main part of the paper is Sect. 3, where we introduce wave function renormalization not by changing the normalization of an elementary field but by changing that of  $K_\Lambda$ . For keeping Sect. 3 short and clear as much as possible, we have collected all the relevant (mostly known but some new) results in Sect. 2. Almost everything we write in Sect. 2 has been written before, but it is handy to have them all here. This section also serves the purpose of making the reader familiar with our notation. A well-informed reader can skip this section. In Sect. 4 we rewrite the ERG differential equations obtained in Sect. 3 using the dimensionless notation. This is necessary to have fixed point solutions. We will be brief since the rewriting is pretty standard. In Sect. 5 we summarize and conclude the paper. We have added four appendices. Appendix A supplements further the review given in Sect. 2. We hope this makes the paper self-contained without getting too long. We then explain the relation of our results to those obtained by C. Bervillier [16–18] in Appendix B and those by Osborn and Twigg [13] and by Rosten [14] in Appendix C. Finally, in Appendix D, we give examples of calculating the anomalous dimensions using ERG perturbatively.

## 2 Mostly review of the relevant results

There are many reviews available on the subject of ERG [1–9]. Rather than referring the readers to them, we review all the necessary results in this preparatory section. Parts of 2.3–5 are new for the viewpoint we provide. Here is a word of warning: we have adopted a not-so-popular convention for the sign of the Wilson action  $S_\Lambda$  so that the Boltzmann weight is  $e^{S_\Lambda}$  instead of the more usual  $e^{-S_\Lambda}$ . This reduces the number of minus signs in the formulas. We work in  $D$  dimensional Euclidean space, and we use the notation

$$\int_p = \int \frac{d^D p}{(2\pi)^D}, \quad \delta(p) = (2\pi)^D \delta^{(D)}(p) \quad (1)$$

for momentum integrals and the  $D$ -dimensional delta function.

### 2.1 ERG differential equation

Let  $S_\Lambda$  be a Wilson action with a momentum cutoff  $\Lambda$ . As we lower  $\Lambda$ , we change  $S_\Lambda$  such that physics is preserved. The specific  $\Lambda$ -dependence is given by the exact renormalization group (ERG) differential equation

$$\begin{aligned} -\Lambda \frac{\partial S_\Lambda[\phi]}{\partial \Lambda} &= \int_p \Lambda \frac{\partial \ln K_\Lambda(p)}{\partial \Lambda} \phi(p) \frac{\delta S_\Lambda[\phi]}{\delta \phi(p)} \\ &+ \int_p \Lambda \frac{\partial}{\partial \Lambda} \ln \frac{K_\Lambda(p)^2}{k_\Lambda(p)} \cdot \frac{k_\Lambda(p)}{p^2} \frac{1}{2} \left\{ \frac{\delta S_\Lambda[\phi]}{\delta \phi(p)} \frac{\delta S_\Lambda[\phi]}{\delta \phi(-p)} + \frac{\delta^2 S_\Lambda[\phi]}{\delta \phi(p) \delta \phi(-p)} \right\}, \end{aligned} \quad (2)$$

which is characterized by two positive cutoff functions  $K_\Lambda(p)$  and  $k_\Lambda(p)$ . [19]  $K_\Lambda(p)$  must have the properties

$$K_\Lambda(p) \longrightarrow \begin{cases} \text{const} & (p^2 \rightarrow 0), \\ 0 & (p^2 \rightarrow \infty), \end{cases} \quad (3)$$

while  $k_\Lambda(p)$  must vanish as  $p^2 \rightarrow 0$ :

$$k_\Lambda(p) \xrightarrow{p^2 \rightarrow 0} 0. \quad (4)$$

Any ERG differential equation given in the past can be written as above if the cutoff functions are chosen appropriately. As we will see in Sect. 3, this is still the case even with an anomalous dimension.

For example, the choice

$$K_\Lambda(p) = K_\Lambda^W(p) \equiv e^{-p^2/\Lambda^2}, \quad k_\Lambda(p) = k_\Lambda^W(p) \equiv \frac{p^2}{\Lambda^2}, \quad (5)$$

was made when the ERG differential equation was written for the first time [20]. For perturbative applications [21], it is convenient to choose

$$K_\Lambda(p) = K(p/\Lambda), \quad K(0) = 1, \quad k_\Lambda(p) = k_\Lambda^P(p) \equiv K(p/\Lambda) (1 - K(p/\Lambda)) \quad (6)$$

so that  $K(p/\Lambda)/p^2$  and  $(1 - K(p/\Lambda))/p^2$  are interpreted as low and high momentum propagators, respectively.

## 2.2 Modified correlation functions

Using  $K_\Lambda$  &  $k_\Lambda$  we can introduce modified correlation functions as

$$\begin{aligned} & \langle\langle \phi(p_1) \cdots \phi(p_n) \rangle\rangle_{S_\Lambda}^{K_\Lambda, k_\Lambda} \\ & \equiv \prod_{i=1}^n \frac{1}{K_\Lambda(p_i)} \cdot \left\langle \exp \left( -\frac{1}{2} \int_p \frac{k_\Lambda(p)}{p^2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right) \phi(p_1) \cdots \phi(p_n) \right\rangle_{S_\Lambda} \\ & = \prod_{i=1}^n \frac{1}{K_\Lambda(p_i)} \cdot \int [d\phi] e^{S_\Lambda[\phi]} \\ & \quad \times \exp \left( -\frac{1}{2} \int_p \frac{k_\Lambda(p)}{p^2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right) \{ \phi(p_1) \cdots \phi(p_n) \}. \end{aligned} \quad (7)$$

$S_\Lambda$  is expected to suppress the fluctuations of high momentum modes, and the division by  $K_\Lambda(p)$  enhances those fluctuations. The propagator  $k_\Lambda(p)/p^2$  modifies two-point functions only at momenta of order  $\Lambda$  or higher. The modified correlation functions thus defined are independent of the cutoff  $\Lambda$  if  $S_\Lambda$  satisfies (2). In fact, as shown in [19], we can derive the ERG differential equation (2) by demanding the  $\Lambda$ -independence of (7).

In the first half of Appendix A we solve (2) and show the  $\Lambda$  independence of (7).

## 2.3 Equivalent Wilson actions

Let  $K'_\Lambda$  be a cutoff function alternative to  $K_\Lambda$ . Substituting  $(K_\Lambda(p)/K'_\Lambda(p)) \phi(p)$  for  $\phi(p)$  in the above modified correlation functions, we obtain

$$\prod_{i=1}^n \frac{1}{K'_\Lambda(p_i)} \cdot \left\langle \exp \left( -\frac{1}{2} \int_p \frac{k_\Lambda(p)}{p^2} \frac{K'_\Lambda(p)^2}{K_\Lambda(p)^2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right) \phi(p_1) \cdots \phi(p_n) \right\rangle_{S'_\Lambda} \quad (8)$$

where  $S'_\Lambda[\phi]$  is the Wilson action obtained by the substitution:

$$S'_\Lambda[\phi] \equiv S_\Lambda \left[ \frac{K_\Lambda(p)}{K'_\Lambda(p)} \phi(p) \right]. \quad (9)$$

If we define

$$k'_\Lambda(p) \equiv k_\Lambda(p) \frac{K'_\Lambda(p)^2}{K_\Lambda(p)^2}, \quad (10)$$

then  $S_\Lambda$  with  $K_\Lambda, k_\Lambda$  has the same modified correlation functions as  $S'_\Lambda$  with  $K'_\Lambda, k'_\Lambda$ :

$$\langle\langle \phi(p_1) \cdots \phi(p_n) \rangle\rangle_{S'_\Lambda}^{K'_\Lambda, k'_\Lambda} = \langle\langle \phi(p_1) \cdots \phi(p_n) \rangle\rangle_{S_\Lambda}^{K_\Lambda, k_\Lambda}. \quad (11)$$

These are independent of  $\Lambda$  if  $S_\Lambda$  satisfies the ERG differential equation (2) with  $K_\Lambda, k_\Lambda$ . Then,  $S'_\Lambda$  is guaranteed to satisfy the ERG differential equation with  $K'_\Lambda, k'_\Lambda$ . We can regard the combination  $(S'_\Lambda, K'_\Lambda, k'_\Lambda)$  as equivalent to the combination  $(S_\Lambda, K_\Lambda, k_\Lambda)$ : equivalent actions give the same modified correlation functions.

Please note that a somewhat broader definition of equivalence was introduced in [19]: a combination  $(S, K, k)$  of a Wilson action and two cutoff functions is regarded as equivalent to  $(S', K', k')$  if they give the same modified correlation functions. Hence,  $(S_\Lambda, K_\Lambda, k_\Lambda)$  for all  $\Lambda$  belongs to the same class of equivalent actions. (Strictly speaking,  $\Lambda$  should not be smaller than the physical mass.) We have adopted a narrower definition so that two equivalent actions are related to each other by a simple linear field transformation (9).

#### 2.4 Functional $W_\Lambda[J]$

Given  $S_\Lambda$  satisfying (2) with  $K_\Lambda, k_\Lambda$ , we define

$$\tilde{S}_\Lambda[\phi] \equiv \frac{1}{2} \int_p \frac{p^2}{k_\Lambda(p)} \phi(p) \phi(-p) + S_\Lambda[\phi]. \quad (12)$$

This satisfies

$$\begin{aligned} -\Lambda \frac{\partial \tilde{S}_\Lambda[\phi]}{\partial \Lambda} &= \int_p \Lambda \frac{\partial \ln \frac{K_\Lambda(p)}{R_\Lambda(p)}}{\partial \Lambda} \phi(p) \frac{\delta \tilde{S}_\Lambda[\phi]}{\delta \phi(p)} \\ &+ \int_p \Lambda \frac{\partial R_\Lambda(p)}{\partial \Lambda} \frac{1}{2} \frac{K_\Lambda(p)^2}{R_\Lambda(p)^2} \left\{ \frac{\delta \tilde{S}_\Lambda[\phi]}{\delta \phi(p)} \frac{\delta \tilde{S}_\Lambda[\phi]}{\delta \phi(-p)} + \frac{\delta^2 \tilde{S}_\Lambda[\phi]}{\delta \phi(p) \delta \phi(-p)} \right\}, \end{aligned} \quad (13)$$

where we define

$$R_\Lambda(p) \equiv \frac{p^2}{k_\Lambda(p)} K_\Lambda(p)^2. \quad (14)$$

The first term can be eliminated by the change of field variables from  $\phi(p)$  to

$$J(p) \equiv \frac{R_\Lambda(p)}{K_\Lambda(p)} \phi(p). \quad (15)$$

Defining

$$\begin{aligned} W_\Lambda[J] &\equiv \tilde{S}_\Lambda \left[ \frac{K_\Lambda(p)}{R_\Lambda(p)} J(p) \right] \\ &= \frac{1}{2} \int_p J(p) \frac{1}{R_\Lambda(p)} J(-p) + S_\Lambda \left[ \frac{K_\Lambda(p)}{R_\Lambda(p)} J(p) \right], \end{aligned} \quad (16)$$

we obtain

$$-\Lambda \frac{\partial W_\Lambda[J]}{\partial \Lambda} = \frac{1}{2} \int_p \Lambda \frac{\partial R_\Lambda(p)}{\partial \Lambda} \left\{ \frac{\delta W_\Lambda[J]}{\delta J(p)} \frac{\delta W_\Lambda[J]}{\delta J(-p)} + \frac{\delta^2 W_\Lambda[J]}{\delta J(p) \delta J(-p)} \right\}, \quad (17)$$

which depends only on the single combination  $R_\Lambda(p)$  of the two cutoff functions. This equation was first obtained by T. Morris in [10, 11].

Given  $W_\Lambda[J]$  with a choice of  $R_\Lambda(p)$ , the corresponding Wilson action  $S_\Lambda[\phi]$  is not uniquely determined. To specify  $S_\Lambda[\phi]$  we need to specify one more cutoff function, say  $K_\Lambda(p)$ . Then,  $k_\Lambda(p)$  is given by

$$k_\Lambda(p) = \frac{p^2}{R_\Lambda(p)} K_\Lambda(p)^2, \quad (18)$$

and  $S_\Lambda[\phi]$  is determined as

$$S_\Lambda[\phi] = -\frac{1}{2} \int_p \frac{p^2}{k_\Lambda(p)} \phi(p) \phi(-p) + W_\Lambda \left[ \frac{R_\Lambda(p)}{K_\Lambda(p)} \phi(p) \right]. \quad (19)$$

If we choose  $K'_\Lambda(p)$  instead of  $K_\Lambda(p)$ , the resulting  $S'_\Lambda$  and  $k'_\Lambda$  satisfy respectively (9) and (10). Hence, the pair  $(W_\Lambda, R_\Lambda)$  corresponds to a class of equivalent Wilson actions  $(S_\Lambda, K_\Lambda, k_\Lambda), (S'_\Lambda, K'_\Lambda, k'_\Lambda), \dots$  all of which give rise to the same modified correlation functions.

### 2.5 Legendre transformation

Given  $W_\Lambda[J]$ , we introduce a Legendre transformation:

$$\tilde{\Gamma}_\Lambda[\Phi] = W_\Lambda[J] - \int_p J(-p) \Phi(p), \quad (20)$$

where, for given  $\Phi(p)$ , we determine  $J(p)$  by

$$\Phi(p) = \frac{\delta W_\Lambda[J]}{\delta J(-p)}. \quad (21)$$

The inverse transformation is given by (20) where, for given  $J(p)$ ,  $\Phi(p)$  is determined by

$$J(p) = -\frac{\delta \tilde{\Gamma}_\Lambda[\Phi]}{\delta \Phi(-p)}. \quad (22)$$

Let  $(S_\Lambda, K_\Lambda, k_\Lambda)$  be one of the combinations corresponding to  $W_\Lambda[J]$ . We can then rewrite (21) as

$$\Phi(p) = \frac{1}{K_\Lambda(p)} \left( \phi(p) + \frac{k_\Lambda(p)}{p^2} \frac{\delta S_\Lambda[\phi]}{\delta \phi(-p)} \right). \quad (23)$$

This is a composite operator corresponding to the elementary field  $\phi(p)$ . Its modified correlation functions satisfy

$$\begin{aligned}
& \langle\langle \Phi(p)\phi(p_1)\cdots\phi(p_n) \rangle\rangle_{S_\Lambda}^{K_\Lambda, k_\Lambda} \\
& \equiv \prod_{i=1}^n \frac{1}{K_\Lambda(p_i)} \cdot \left\langle \Phi(p) \exp \left( -\frac{1}{2} \int_p \frac{k_\Lambda(p)}{p^2} \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right) \{ \phi(p_1)\cdots\phi(p_n) \} \right\rangle_{S_\Lambda} \\
& = \langle\langle \phi(p)\phi(p_1)\cdots\phi(p_n) \rangle\rangle_{S_\Lambda}^{K_\Lambda, k_\Lambda} .
\end{aligned} \tag{24}$$

It is a general property of the Legendre transformation that the second order differentials

$$\frac{\delta W_\Lambda[J]}{\delta J(p)\delta J(-q)} = \frac{\delta\Phi(-p)}{\delta J(-q)}, \quad (-) \frac{\delta\tilde{\Gamma}_\Lambda[\Phi]}{\delta\Phi(p)\delta\Phi(-q)} = \frac{\delta J(-p)}{\delta\Phi(-q)} \tag{25}$$

are the inverse of each other:

$$\int_q \frac{\delta^2 W_\Lambda[J]}{\delta J(p)\delta J(-q)} (-) \frac{\delta^2 \tilde{\Gamma}_\Lambda[\Phi]}{\delta\Phi(q)\delta\Phi(-r)} = \delta(p-r) . \tag{26}$$

We will use the notation

$$G_{\Lambda;p,-q}[\Phi] \equiv \frac{\delta^2 W_\Lambda[J]}{\delta J(p)\delta J(-q)} \tag{27}$$

when we prefer to regard this as a functional of  $\Phi$ .

Another general property of the Legendre transformation is that  $W_\Lambda[J]$  and  $\tilde{\Gamma}_\Lambda[\Phi]$  share the same  $\Lambda$ -dependence:

$$\begin{aligned}
-\Lambda \frac{\partial \tilde{\Gamma}_\Lambda[\Phi]}{\partial \Lambda} &= -\Lambda \frac{\partial W_\Lambda[J]}{\partial \Lambda} \\
&= \frac{1}{2} \int_p \Lambda \frac{\partial R_\Lambda(p)}{\partial \Lambda} \left\{ \Phi(p)\Phi(-p) + \frac{\delta^2 W_\Lambda[J]}{\delta J(p)\delta J(-p)} \right\} \\
&= \frac{1}{2} \int_p \Lambda \frac{\partial R_\Lambda(p)}{\partial \Lambda} \left\{ \Phi(p)\Phi(-p) + G_{\Lambda;p,-p}[\Phi] \right\} ,
\end{aligned} \tag{28}$$

where we have used (17) and (27).

We now define the 1PI action  $\Gamma_\Lambda[\Phi]$  so that

$$\tilde{\Gamma}_\Lambda[\Phi] = -\frac{1}{2} \int_p R_\Lambda(p) \Phi(p)\Phi(-p) + \Gamma_\Lambda[\Phi] . \tag{29}$$

The excluded term is often called a scale dependent mass term. The 1PI action has a very simple  $\Lambda$ -dependence:

$$-\Lambda \frac{\partial \Gamma_\Lambda[\Phi]}{\partial \Lambda} = \frac{1}{2} \int_p \Lambda \frac{\partial R_\Lambda(p)}{\partial \Lambda} G_{\Lambda;p,-p}[\Phi] . \tag{30}$$



Using  $\Gamma_\Lambda$  we can rewrite (26) as

$$\int_q G_{\Lambda;p,-q}[\Phi] \left( R_\Lambda(q-r)\delta(q-r) - \frac{\delta^2 \Gamma_\Lambda[\Phi]}{\delta \Phi(q) \delta \Phi(-r)} \right) = \delta(p-r). \quad (31)$$

It is important to note that the 1PI action  $\Gamma_\Lambda[\Phi]$  (with  $R_\Lambda$ ) corresponds one-to-one to  $W_\Lambda[J]$  (with  $R_\Lambda$ ). Hence, all the equivalent combinations  $(S_\Lambda, K_\Lambda, k_\Lambda)$ , giving rise to the same modified correlation functions, correspond to the same 1PI action. We end this section by writing down the Legendre transformation (20) using  $S_\Lambda$  instead of  $W_\Lambda$ :

$$-\frac{1}{2} \int_p R_\Lambda(p) \Phi(p) \Phi(-p) + \Gamma_\Lambda[\Phi] = \frac{1}{2} \int_p \frac{p^2}{k_\Lambda(p)} \phi(p) \phi(-p) + S_\Lambda[\phi] - \int_p \frac{R_\Lambda(p)}{K_\Lambda(p)} \phi(-p) \Phi(p). \quad (32)$$

### 3 Wave function renormalization for the Wilson and 1PI actions

After a long preparation, we are ready to discuss wave function renormalization in the ERG formalism. We first introduce a cutoff dependent positive wave function renormalization constant  $Z_\Lambda$ . We denote the anomalous dimension by

$$\gamma_\Lambda = -\Lambda \frac{\partial}{\partial \Lambda} \ln \sqrt{Z_\Lambda}. \quad (33)$$

Physics dictates an appropriate choice of  $Z_\Lambda$ . For now, we can keep it arbitrary. We wish to construct a Wilson action whose modified correlation functions are proportional to appropriate powers of  $Z_\Lambda$ :

$$\langle\langle \phi(p_1) \cdots \phi(p_n) \rangle\rangle_{S_\Lambda}^{K_\Lambda, k_\Lambda} = Z_\Lambda^{\frac{n}{2}} \cdot (\Lambda\text{-independent}). \quad (34)$$

This implies

$$\frac{1}{Z_\Lambda^{\frac{n}{2}}} \langle\langle \phi(p_1) \cdots \phi(p_n) \rangle\rangle_{S_\Lambda}^{K_\Lambda, k_\Lambda} = \langle\langle \phi(p_1) \cdots \phi(p_n) \rangle\rangle_{S_\Lambda}^{\sqrt{Z_\Lambda} K_\Lambda, k_\Lambda} = (\Lambda\text{-independent}). \quad (35)$$

Hence,  $S_\Lambda$  satisfies the ERG differential equation for the cutoff functions  $K_\Lambda^Z \equiv \sqrt{Z_\Lambda} K_\Lambda$  and  $k_\Lambda$ :

$$\begin{aligned}
-\Lambda \frac{\partial S_\Lambda[\phi]}{\partial \Lambda} &= \int_p \Lambda \frac{\partial \ln(\sqrt{Z_\Lambda} K_\Lambda(p))}{\partial \Lambda} \phi(p) \frac{\delta S_\Lambda[\phi]}{\delta \phi(p)} \\
&+ \int_p \Lambda \frac{\partial}{\partial \Lambda} \ln \frac{Z_\Lambda K_\Lambda(p)^2}{k_\Lambda(p)} \cdot \frac{k_\Lambda(p)}{p^2} \frac{1}{2} \left\{ \frac{\delta S_\Lambda[\phi]}{\delta \phi(p)} \frac{\delta S_\Lambda[\phi]}{\delta \phi(-p)} + \frac{\delta^2 S_\Lambda[\phi]}{\delta \phi(p) \delta \phi(-p)} \right\} \\
&= \int_p \Lambda \frac{\partial \ln K_\Lambda(p)}{\partial \Lambda} \phi(p) \frac{\delta S_\Lambda[\phi]}{\delta \phi(p)} \\
&+ \int_p \Lambda \frac{\partial}{\partial \Lambda} \ln \frac{K_\Lambda(p)^2}{k_\Lambda(p)} \cdot \frac{k_\Lambda(p)}{p^2} \frac{1}{2} \left\{ \frac{\delta S_\Lambda[\phi]}{\delta \phi(p)} \frac{\delta S_\Lambda[\phi]}{\delta \phi(-p)} + \frac{\delta^2 S_\Lambda[\phi]}{\delta \phi(p) \delta \phi(-p)} \right\} \\
&- \gamma_\Lambda \int_p \left[ \phi(p) \frac{\delta S_\Lambda[\phi]}{\delta \phi(p)} + \frac{k_\Lambda(p)}{p^2} \left\{ \frac{\delta S_\Lambda[\phi]}{\delta \phi(p)} \frac{\delta S_\Lambda[\phi]}{\delta \phi(-p)} + \frac{\delta^2 S_\Lambda[\phi]}{\delta \phi(p) \delta \phi(-p)} \right\} \right]. \quad (36)
\end{aligned}$$

This is the general form of the ERG differential equation with an anomalous dimension. (This result is not unknown; its dimensionless form was given in [19] as (43).)

The last term proportional to  $\gamma_\Lambda$  is worthy of a comment. It is an equation-of-motion composite operator that counts the number of  $\phi$ 's:

$$\mathcal{N}_\Lambda[\phi] \equiv - \int_p K_\Lambda(p) e^{-S_\Lambda} \frac{\delta}{\delta \phi(p)} \left[ \Phi(p) e^{S_\Lambda} \right],$$

where  $\Phi(p)$  is defined by (23). Using (23), we obtain

$$\begin{aligned}
\mathcal{N}_\Lambda[\phi] &= - \int_p e^{-S_\Lambda} \frac{\delta}{\delta \phi(p)} \left[ \left( \phi(p) + \frac{k_\Lambda(p)}{p^2} \frac{\delta S_\Lambda}{\delta \phi(-p)} \right) e^{S_\Lambda} \right] \\
&= - \int_p \left[ \phi(p) \frac{\delta S_\Lambda[\phi]}{\delta \phi(p)} + \frac{k_\Lambda(p)}{p^2} \left\{ \frac{\delta S_\Lambda[\phi]}{\delta \phi(p)} \frac{\delta S_\Lambda[\phi]}{\delta \phi(-p)} + \frac{\delta^2 S_\Lambda[\phi]}{\delta \phi(p) \delta \phi(-p)} \right\} \right] \quad (37)
\end{aligned}$$

up to an additive field independent constant. This has the modified correlation functions

$$\begin{aligned}
&\llbracket \mathcal{N}_\Lambda \phi(p_1) \cdots \phi(p_n) \rrbracket_{S_\Lambda}^{K_\Lambda, k_\Lambda} \\
&\equiv \prod_{i=1}^n \frac{1}{K_\Lambda(p_i)} \cdot \left\langle \mathcal{N}_\Lambda \exp \left( -\frac{1}{2} \int_p \frac{k_\Lambda(p)}{p^2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right) \{ \phi(p_1) \cdots \phi(p_n) \} \right\rangle_{S_\Lambda} \\
&= n \llbracket \phi(p_1) \cdots \phi(p_n) \rrbracket_{S_\Lambda}^{K_\Lambda, k_\Lambda}. \quad (38)
\end{aligned}$$

$\mathcal{N}_\Lambda$  is a particular example of equation-of-motion composite operators, also called redundant operators, marginal operators, or exactly marginal redundant operators in the literature. (See [22] for the original discussion. In the context of ERG, see, for example, [23], [24], Appendix A of [13], and the reviews [8, 9].)

Using the prescription given in Sect. 2.5, we can construct a 1PI action. The result depends on whether we take  $K_\Lambda$  or  $K_\Lambda^Z = \sqrt{Z_\Lambda} K_\Lambda$  as one of the cutoff functions together with the fixed  $k_\Lambda$ . Let us first consider the combination  $(S_\Lambda, K_\Lambda, k_\Lambda)$  that corresponds to  $Z_\Lambda$ -dependent modified correlation functions. We then obtain

$$W_\Lambda[J] \equiv \tilde{S}_\Lambda \left[ \frac{K_\Lambda(p)}{R_\Lambda(p)} J(p) \right] \quad (39)$$

and the 1PI action

$$-\frac{1}{2} \int_p R_\Lambda(p) \Phi(p) \Phi(-p) + \Gamma_\Lambda[\Phi] = W_\Lambda[J] - \int_p J(-p) \Phi(p). \quad (40)$$

We repeat the above for the combination  $(S_\Lambda, K_\Lambda^Z, k_\Lambda)$  that corresponds to  $\Lambda$ -independent modified correlation functions. Since

$$R_\Lambda^Z(p) \equiv \frac{p^2}{k_\Lambda(p)} K_\Lambda^Z(p)^2 = Z_\Lambda R_\Lambda(p), \quad (41)$$

we obtain

$$W_\Lambda^Z[J] \equiv \tilde{S}_\Lambda \left[ \frac{K_\Lambda^Z(p)}{R_\Lambda^Z(p)} J(p) \right] = \tilde{S}_\Lambda \left[ \frac{K_\Lambda(p)}{R_\Lambda(p)} \frac{J(p)}{\sqrt{Z_\Lambda}} \right] = W_\Lambda \left[ \frac{J}{\sqrt{Z_\Lambda}} \right] \quad (42)$$

and

$$\begin{aligned} -\frac{1}{2} \int_p Z_\Lambda R_\Lambda(p) \Phi(p) \Phi(-p) + \Gamma_\Lambda^Z[\Phi] &= W_\Lambda^Z[J] - \int_p J(-p) \Phi(p) \\ &= W_\Lambda \left[ \frac{J}{\sqrt{Z_\Lambda}} \right] - \int_p \frac{J(-p)}{\sqrt{Z_\Lambda}} \sqrt{Z_\Lambda} \Phi(p). \end{aligned} \quad (43)$$

Comparing this with (40), we obtain

$$\Gamma_\Lambda^Z[\Phi] = \Gamma_\Lambda \left[ \sqrt{Z_\Lambda} \Phi \right]. \quad (44)$$

Please note that the same 1PI action is obtained from any combination  $(S'_\Lambda, K'_\Lambda, k'_\Lambda)$  equivalent to  $(S_\Lambda, K_\Lambda^Z, k_\Lambda)$ . For example, with the choice  $K'_\Lambda = K_\Lambda, k'_\Lambda = k_\Lambda/Z_\Lambda$ , the Wilson action

$$S'_\Lambda[\phi] = S_\Lambda \left[ \sqrt{Z_\Lambda} \phi \right] \quad (45)$$

gives the same  $\Gamma_\Lambda^Z[\Phi]$ .

Let us find the  $\Lambda$ -dependence of the 1PI actions. The prescription of Sect. 2.5 applies directly to  $\Gamma_\Lambda^Z[\Phi]$  that corresponds to  $\Lambda$ -independent modified correlation functions. We

obtain

$$-\Lambda \frac{\partial \Gamma_\Lambda^Z[\Phi]}{\partial \Lambda} = \frac{1}{2} \int_p \Lambda \frac{\partial (Z_\Lambda R_\Lambda(p))}{\partial \Lambda} G_{\Lambda;p,-p}^Z[\Phi], \quad (46)$$

where

$$\int_q G_{\Lambda;p,-q}^Z[\Phi] \left( Z_\Lambda R_\Lambda(q) \delta(q-r) - \frac{\delta^2 \Gamma_\Lambda^Z[\Phi]}{\delta \Phi(q) \delta \Phi(-r)} \right) = \delta(p-r). \quad (47)$$

This result for  $\Gamma_\Lambda^Z$  is given in [12] for gauge theories and in [5] for generic scalar theories. Consequently, the ERG differential equation for

$$\Gamma_\Lambda[\Phi] = \Gamma_\Lambda^Z \left[ \frac{\Phi}{\sqrt{Z_\Lambda}} \right] \quad (48)$$

is obtained as

$$-\Lambda \frac{\partial \Gamma_\Lambda[\Phi]}{\partial \Lambda} = -\gamma_\Lambda \int_p \Phi(p) \frac{\delta \Gamma_\Lambda[\Phi]}{\delta \Phi(p)} + \frac{1}{2} \int_p \left( \frac{\partial R_\Lambda(p)}{\partial \Lambda} - 2\gamma_\Lambda R_\Lambda(p) \right) G_{\Lambda;p,-p}[\Phi], \quad (49)$$

where

$$G_{\Lambda;p,-q}[\Phi] \equiv Z_\Lambda G_{\Lambda;p,-q}^Z \left[ \frac{\Phi}{\sqrt{Z_\Lambda}} \right] \quad (50)$$

satisfies

$$\int_q G_{\Lambda;p,-q}[\Phi] \left( R_\Lambda(q) \delta(q-r) - \frac{\delta^2 \Gamma_\Lambda[\Phi]}{\delta \Phi(q) \delta \Phi(-r)} \right) = \delta(p-r). \quad (51)$$

We regard (49) for  $\Gamma_\Lambda$ , first obtained by T. Morris in [10, 11], as the general form of the ERG differential equation for the 1PI actions.

In (49), the term proportional to  $\gamma_\Lambda$  is given by the equation-of-motion operator

$$\mathcal{N}_\Lambda^{1\text{PI}}[\Phi] \equiv - \int_p \left( \Phi(p) \frac{\delta \Gamma_\Lambda[\Phi]}{\delta \Phi(p)} + R_\Lambda(p) G_{\Lambda;p,-p}[\Phi] \right). \quad (52)$$

This equals  $\mathcal{N}_\Lambda[\phi]$  given by (37), written in terms of  $\Phi$  instead of  $\phi$ .

## 4 Fixed point

So far we have considered an arbitrary anomalous dimension  $\gamma_\Lambda$  and its integral  $Z_\Lambda$ . Its introduction becomes essential when we look for a fixed point of the ERG differential equation, either for the Wilson action or the 1PI action. For the differential equation to have a fixed point, we need to adopt the dimensionless notation by measuring dimensionful quantities in units of appropriate powers of the momentum cutoff  $\Lambda$ . Rewriting of the

ERG differential equation (36) and (49,51) is straightforward. We only give results here. We introduce a logarithmic scale parameter  $t$  by

$$\Lambda = \mu e^{-t}, \quad (53)$$

where  $\mu$  is an arbitrary momentum scale. A different choice of  $\mu$  amounts to a constant shift of  $t$ . Denoting the cutoff functions by

$$K_\Lambda(p) = K_t(p/\Lambda), \quad k_\Lambda(p) = k_t(p/\Lambda), \quad (54)$$

we can rewrite (36) as

$$\begin{aligned} \partial_t S_t[\phi] = & \int_p \left[ \left\{ \left( -\partial_t - p_\mu \frac{\partial}{\partial p_\mu} \right) \ln K_t(p) + \frac{D+2}{2} \right\} \phi(p) + p_\mu \frac{\partial \phi(p)}{\partial p_\mu} \right] \frac{\delta S_t[\phi]}{\delta \phi(p)} \\ & + \int_p \left( \partial_t + p_\mu \frac{\partial}{\partial p_\mu} \right) \ln \frac{k_t(p)}{K_t(p)^2} \cdot \frac{k_t(p)}{p^2} \frac{1}{2} \left\{ \frac{\delta S_t[\phi]}{\delta \phi(p)} \frac{\delta S_t[\phi]}{\delta \phi(-p)} + \frac{\delta^2 S_t[\phi]}{\delta \phi(p) \delta \phi(-p)} \right\} \\ & - \gamma_t \int_p \left[ \phi(p) \frac{\delta S_t[\phi]}{\delta \phi(p)} + \frac{k_t(p)}{p^2} \left\{ \frac{\delta S_t[\phi]}{\delta \phi(p)} \frac{\delta S_t[\phi]}{\delta \phi(-p)} + \frac{\delta^2 S_t[\phi]}{\delta \phi(p) \delta \phi(-p)} \right\} \right], \end{aligned} \quad (55)$$

where we have denoted  $\gamma_\Lambda$  as  $\gamma_t$ . Similarly, we can rewrite (49) as

$$\begin{aligned} \partial_t \Gamma_t[\Phi] = & \int_p \left( \frac{D+2}{2} + p_\mu \frac{\partial}{\partial p_\mu} \right) \Phi(p) \cdot \frac{\delta \Gamma_t[\Phi]}{\delta \Phi(p)} \\ & + \int_p \left( 2 - \left( \partial_t + p_\mu \frac{\partial}{\partial p_\mu} \right) \ln R_t(p) \right) \cdot R_t(p) \frac{1}{2} G_{t;p,-p}[\Phi] \\ & - \gamma_t \int_p \left( \Phi(p) \frac{\delta \Gamma_t[\Phi]}{\delta \Phi(p)} + R_t(p) G_{t;p,-p}[\Phi] \right), \end{aligned} \quad (56)$$

where

$$R_t(p) \equiv \frac{p^2}{k_t(p)} K_t(p)^2, \quad (57)$$

and (51) as

$$\int_q G_{t;p,-q}[\Phi] \left\{ R_t(q) \delta(q-r) - \frac{\delta^2 \Gamma_t[\Phi]}{\delta \Phi(q) \delta \Phi(-r)} \right\} = \delta(p-r). \quad (58)$$

To obtain a fixed point, we must choose  $t$ -independent cutoff functions:

$$\begin{cases} K_t(p) = K(p), \\ k_t(p) = k(p). \end{cases} \quad (59)$$

Then, the above ERG differential equations become simpler:

$$\begin{aligned} \partial_t S_t[\phi] = & \int_p \left\{ -p_\mu \frac{\partial}{\partial p_\mu} \ln K(p) + \frac{D+2}{2} - \gamma_t + p_\mu \frac{\partial}{\partial p_\mu} \right\} \phi(p) \cdot \frac{\delta S_t[\phi]}{\delta \phi(p)} \\ & + \int_p \left\{ -p_\mu \frac{\partial}{\partial p_\mu} \ln R(p) + 2 - 2\gamma_t \right\} \frac{k(p)}{p^2} \frac{1}{2} \left\{ \frac{\delta S_t[\phi]}{\delta \phi(p)} \frac{\delta S_t[\phi]}{\delta \phi(-p)} + \frac{\delta^2 S_t[\phi]}{\delta \phi(p) \delta \phi(-p)} \right\}, \end{aligned} \quad (60)$$

and

$$\begin{aligned}\partial_t \Gamma_t[\Phi] &= \int_p \left( \frac{D+2}{2} - \gamma_t + p_\mu \frac{\partial}{\partial p_\mu} \right) \Phi(p) \cdot \frac{\delta \Gamma_t[\Phi]}{\delta \Phi(p)} \\ &\quad + \int_p \left( -p_\mu \frac{\partial}{\partial p_\mu} \ln R(p) + 2 - 2\gamma_t \right) \cdot R(p) \frac{1}{2} G_{t;p,-p}[\Phi],\end{aligned}\quad (61)$$

where

$$R(p) \equiv \frac{p^2}{k(p)} K(p)^2, \quad (62)$$

and  $G_{t;p,-q}[\Phi]$  is defined by

$$\int_q G_{t;p,-q}[\Phi] \left\{ R(q) \delta(q-r) - \frac{\delta^2 \Gamma_t[\Phi]}{\delta \Phi(q) \delta \Phi(-r)} \right\} = \delta(p-r). \quad (63)$$

The anomalous dimension  $\gamma_t$  can be chosen so as to fix a particular term (the kinetic term, for example) in  $S_t$ . Alternatively, it can be chosen as the fixed-point value  $\gamma^*$  in a neighborhood of the fixed point.

The fixed point action  $S^*$  satisfies

$$\begin{aligned}0 &= \int_p \left\{ -p_\mu \frac{\partial}{\partial p_\mu} \ln K(p) + \frac{D+2}{2} - \gamma^* + p_\mu \frac{\partial}{\partial p_\mu} \right\} \phi(p) \cdot \frac{\delta S^*[\phi]}{\delta \phi(p)} \\ &\quad + \int_p \left\{ -p_\mu \frac{\partial}{\partial p_\mu} \ln R(p) + 2 - 2\gamma^* \right\} \frac{k(p)}{p^2} \frac{1}{2} \left\{ \frac{\delta S^*[\phi]}{\delta \phi(p)} \frac{\delta S^*[\phi]}{\delta \phi(-p)} + \frac{\delta^2 S^*[\phi]}{\delta \phi(p) \delta \phi(-p)} \right\},\end{aligned}\quad (64)$$

and the corresponding 1PI action  $\Gamma^*$  satisfies

$$\begin{aligned}0 &= \int_p \left( \frac{D+2}{2} - \gamma^* + p_\mu \frac{\partial}{\partial p_\mu} \right) \Phi(p) \cdot \frac{\delta \Gamma^*[\Phi]}{\delta \Phi(p)} \\ &\quad + \int_p \left( -p_\mu \frac{\partial}{\partial p_\mu} \ln R(p) + 2 - 2\gamma^* \right) \cdot R(p) \frac{1}{2} G_{p,-p}^*[\Phi],\end{aligned}\quad (65)$$

where

$$\int_q G_{p,-q}^*[\Phi] \left\{ R(q) \delta(q-r) - \frac{\delta^2 \Gamma^*[\Phi]}{\delta \Phi(q) \delta \Phi(-r)} \right\} = \delta(p-r). \quad (66)$$

We can solve (64) and (65) only for particular choices of  $\gamma^*$ .

## 5 Summary and conclusions

In this paper we have made the best effort to elucidate the structure of the exact renormalization group both for the Wilson actions and for the 1PI actions. Especially, we have tried to demonstrate the simplicity of introducing an anomalous dimension to the ERG differential equations.

We have started with introducing classes of equivalent Wilson actions. A Wilson action  $S_\Lambda$  with a momentum cutoff  $\Lambda$  is paired with two cutoff functions of momentum:  $K_\Lambda(p)$  and  $k_\Lambda(p)$ . We then construct modified correlation functions (7) using  $K_\Lambda$  and  $k_\Lambda$ . A class of equivalent Wilson actions consists of those combinations of  $(S_\Lambda, K_\Lambda, k_\Lambda)$  giving the same modified correlation functions. The equivalence of  $(S_\Lambda, K_\Lambda, k_\Lambda)$  and  $(S'_\Lambda, K'_\Lambda, k'_\Lambda)$  demands that  $R_\Lambda(p) \equiv \frac{p^2}{k_\Lambda(p)} K_\Lambda(p)^2$  is the same as  $R'_\Lambda(p) \equiv \frac{p^2}{k'_\Lambda(p)} K'_\Lambda(p)^2$ , and that  $S_\Lambda$  and  $S'_\Lambda$  are related by (9).

The crux of the paper is the observation that all the equivalent Wilson actions correspond to the same 1PI action  $\Gamma_\Lambda[\Phi]$  via the Legendre transformation (20) (equivalently (32) or (40)). This correspondence is many-to-one, since the Wilson action depends on two cutoff functions  $K_\Lambda, k_\Lambda$ ; the 1PI action depends only on  $R_\Lambda$ .

We have introduced the anomalous dimension  $\gamma_\Lambda$  of the elementary field by demanding the  $\Lambda$ -dependence of the modified correlation functions as given by (34). From this we have derived the general form (36) of the ERG differential equation for the Wilson action. We regard (36) as the counterpart of the general form (49) for the 1PI action, introduced previously by Morris [10, 11].

As long as two Wilson actions share the same  $R_\Lambda$ , we can transform one Wilson action to another equivalent one just by changing  $K_\Lambda$ . For example, the Wilson action introduced by Bervillier in [16] has an arbitrary cutoff function  $K_\Lambda(p)$  which he took to be the same as  $R_\Lambda(p)$ . By choosing this  $R_\Lambda(p)$  appropriately (its explicit form is given in Appendix C), we can convert Bervillier's ERG differential equation into the one by Ball et al. given in [15].

Though we have discussed a generic scalar theory in this paper, nothing prevents us from introducing anomalous dimensions to the fermionic and gauge fields by extending our results.

## Acknowledgment

The work of Y. I. and K. I. was partially supported by the JSPS grant-in-aid #R2209 and #22540270. The work of H. S. was partially supported by the JSPS grant-in-aid #25400258.

## References

- [1] C. Becchi, [arXiv:hep-th/9607188].
- [2] T. R. Morris, Prog. Theor. Phys. Suppl. **131**, 395-414 (1998) [arXiv:hep-th/9802039].
- [3] K.-I. Aoki, Int. J. Mod. Phys. B **14**, 1249-1326 (2000).
- [4] C. Bagnuls, C. Bervillier, Phys. Rept. **348**, 91-150 (2001) [arXiv:hep-th/0002034].
- [5] J. Berges, N. Tetradis, C. Wetterich, Phys. Repts. **363**, 223-386 (2002) [arXiv:hep-ph/0005122].
- [6] J. M. Pawłowski, Ann. of Phys. **322**, 2831-2915 (2007) [arXiv:hep-th/0512261].
- [7] H. Gies, Lect. Notes Phys. **852**, 287-348 (2012) [arXiv:hep-ph/0611146].
- [8] Y. Igarashi, K. Itoh, H. Sonoda, Prog. Theor. Phys. Suppl. **181**, 1-166 (2010) [arXiv:0909.0327].
- [9] O. J. Rosten, Phys. Rept. **511**, 177-272 (2012) [arXiv:1003.1366].

- [10] T. Morris, Int. J. Mod. Phys. A **9**, 2411–2450 (1994) [[arXiv:hep-ph/9308265](#)].
- [11] T. Morris, Phys. Lett. B **329**, 241–248 (1994) [[arXiv:hep-ph/9403340](#)].
- [12] U. Ellwanger, M. Hirsch, A. Weber, Z. Phys. C **69**, 687–697 (1996) [[arXiv:hep-th/9506019](#)].
- [13] H. Osborn, D. E. Twigg, Ann. Phys. **327**, 29–73 (2012) [[arXiv:1108.5340](#)].
- [14] O. J. Rosten, [[arXiv:1106.2544](#)].
- [15] R. D. Ball, P. E. Haagensen, J. I. Latorre, E. Moreno, Phys. Lett. B **347**, 80–88 (1995) [[arXiv:hep-th/9411122](#)].
- [16] C. Bervillier, Phys. Lett. A **332**, 93–100 (2004) [[arXiv:hep-th/0405025](#)].
- [17] C. Bervillier, Cond. Matt. Phys. **16**, 23003: 1–10 (2013) [[arXiv:1304.4131](#)].
- [18] C. Bervillier, [[arXiv:1405.0791](#)].
- [19] H. Sonoda, Prog. Theor. Exp. Phys. **2015**, 103B01 (2015) [[arXiv:1503.08578](#)].
- [20] K. G. Wilson, J. Kogut, Phys. Rept. **12**, 75–200 (1974).
- [21] J. Polchinski, Nucl. Phys. B **231**, 269–291 (1984).
- [22] F. J. Wegner, J. Phys. C **7**, 2098–2108 (1974).
- [23] H. Sonoda, J. Phys. A **40**, 5733–5750 (2007) [[arXiv:hep-th/0612294](#)].
- [24] J. O’Dwyer, H. Osborn, Ann. Phys. **323**, 1859–1898 (2008) [[arXiv:0708.2697](#)].

## A Supplement to Sect. 2

### A.1 Integration of the ERG differential equation and the modified correlation functions

The ERG differential equation (2) is a generalized diffusion equation, and it admits a Gaussian integral formula. In this appendix, we first solve the cutoff dependence of the Wilson action, and then show that the modified correlation functions (7) are independent of the cutoff.

To begin with, we rewrite (2) as

$$\begin{aligned}
-\Lambda \frac{\partial}{\partial \Lambda} e^{S_\Lambda[K_\Lambda \phi]} &= \int_p \Lambda \frac{\partial \ln \frac{p^2 K_\Lambda(p)^2}{k_\Lambda(p)}}{\partial \Lambda} \cdot \frac{k_\Lambda(p)}{p^2 K_\Lambda(p)^2} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} e^{S_\Lambda[K_\Lambda \phi]} \\
&= - \int_p \Lambda \frac{\partial \frac{1}{R_\Lambda(p)}}{\partial \Lambda} \frac{1}{2} \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} e^{S_\Lambda[K_\Lambda \phi]}, \tag{A1}
\end{aligned}$$

where

$$R_\Lambda(p) \equiv \frac{p^2 K_\Lambda(p)^2}{k_\Lambda(p)}. \tag{A2}$$

Integrating this, we obtain a Gaussian integral formula:

$$e^{S_{\Lambda_2}[K_{\Lambda_2} \phi]} = \exp \left[ \frac{1}{2} \int_p \left( \frac{1}{R_{\Lambda_2}(p)} - \frac{1}{R_{\Lambda_1}(p)} \right) \frac{\delta^2}{\delta \phi(p) \delta \phi(-p)} \right] e^{S_{\Lambda_1}[K_{\Lambda_1} \phi]}. \tag{A3}$$

We then introduce a generating functional

$$\begin{aligned}
Z_\Lambda[J] &\equiv \int [d\phi] \exp \left( S_\Lambda[\phi] + \int_p J(-p) \frac{\phi(p)}{K_\Lambda(p)} \right) \\
&= \int [d\phi] \exp \left( S_\Lambda[K_\Lambda \phi] + \int_p J(-p) \phi(p) \right). \tag{A4}
\end{aligned}$$



Using (A3), we obtain

$$\begin{aligned}
Z_{\Lambda_2}[J] &= \int [d\phi] \exp \left( \int_p J(-p)\phi(p) \right) e^{S_{\Lambda_2}[K_{\Lambda_2}\phi]} \\
&= \int [d\phi] \exp \left( \int_p J(-p)\phi(p) \right) \\
&\quad \times \exp \left[ \frac{1}{2} \int_p \left( \frac{1}{R_{\Lambda_2}(p)} - \frac{1}{R_{\Lambda_1}(p)} \right) \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right] e^{S_{\Lambda_1}[K_{\Lambda_1}\phi]} \\
&= \int [d\phi] e^{S_{\Lambda_1}[K_{\Lambda_1}\phi]} \\
&\quad \times \exp \left[ \frac{1}{2} \int_p \left( \frac{1}{R_{\Lambda_2}(p)} - \frac{1}{R_{\Lambda_1}(p)} \right) \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right] \exp \left( \int_p J(-p)\phi(p) \right) \\
&= \exp \left[ \frac{1}{2} \int_p \left( \frac{1}{R_{\Lambda_2}(p)} - \frac{1}{R_{\Lambda_1}(p)} \right) J(p)J(-p) \right] Z_{\Lambda_1}[J]. \tag{A5}
\end{aligned}$$

We have thus found that

$$\begin{aligned}
&Z_{\Lambda}[J] \exp \left[ -\frac{1}{2} \int_p \frac{1}{R_{\Lambda}(p)} J(p)J(-p) \right] \\
&= \int [d\phi] \exp \left[ S_{\Lambda}[\phi] + \int_p \left( J(-p) \frac{\phi(p)}{K_{\Lambda}(p)} - \frac{1}{2} \frac{k_{\Lambda}(p)}{p^2} \frac{J(p)}{K_{\Lambda}(p)} \frac{J(-p)}{K_{\Lambda}(p)} \right) \right] \tag{A6}
\end{aligned}$$

is independent of  $\Lambda$ , and it generates the modified correlation functions defined by (7):

$$Z_{\Lambda}[J] \exp \left[ -\frac{1}{2} \int_p \frac{1}{R_{\Lambda}(p)} J(p)J(-p) \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{p_1, \dots, p_n} J(p_1) \cdots J(p_n) \langle\langle \phi(p_1) \cdots \phi(p_n) \rangle\rangle_{S_{\Lambda}}^{K_{\Lambda}, k_{\Lambda}}. \tag{A7}$$

## A.2 Two examples of ERG differential equations

### A.2.1 Wilson

When the ERG differential equation was motivated and derived for the first time in [20], the following particular choice was made for the cutoff functions:

$$\begin{cases} K_{\Lambda}(p) = K_{\Lambda}^W(p) \equiv \exp \left( -\frac{p^2}{\Lambda^2} \right), \\ k_{\Lambda}(p) = k_{\Lambda}^W(p) \equiv \frac{p^2}{\Lambda^2}, \\ R_{\Lambda}(p) = R_{\Lambda}^W(p) \equiv \Lambda^2 \exp \left( -2\frac{p^2}{\Lambda^2} \right). \end{cases} \tag{A8}$$

Correspondingly, (36), (49), and (51) become

$$\begin{aligned}
-\Lambda \frac{\partial S_\Lambda}{\partial \Lambda} &= \int_p \left( 2 \frac{p^2}{\Lambda^2} - \gamma_\Lambda \right) \phi \frac{\delta S_\Lambda}{\delta \phi} \\
&\quad + \int_p \left( 1 + 2 \frac{p^2}{\Lambda^2} - \gamma_\Lambda \right) \frac{1}{\Lambda^2} \left\{ \frac{\delta S_\Lambda}{\delta \phi} \frac{\delta S_\Lambda}{\delta \phi} + \frac{\delta^2 S_\Lambda}{\delta \phi \delta \phi} \right\}, \quad (\text{A9})
\end{aligned}$$

$$-\Lambda \frac{\partial \Gamma_\Lambda}{\partial \Lambda} = -\gamma_\Lambda \int_p \Phi \frac{\delta \Gamma_\Lambda}{\delta \Phi} + \int_p \left( 1 - \gamma_\Lambda + 2 \frac{p^2}{\Lambda^2} \right) \Lambda^2 e^{-\frac{2p^2}{\Lambda^2}} G_{\Lambda;p,-p}, \quad (\text{A10})$$

and

$$\int_q G_{\Lambda;p,-q} \left( \Lambda^2 e^{-\frac{2q^2}{\Lambda^2}} \delta(q-r) - \frac{\delta^2 \Gamma_\Lambda}{\delta \Phi(q) \delta \Phi(-r)} \right) = \delta(p-r). \quad (\text{A11})$$

### A.2.2 Polchinski

The following choice was made in [21]:

$$\begin{cases} K_\Lambda(p) = K(p/\Lambda), \\ k_\Lambda(p) = k_\Lambda^P(p) \equiv K(p/\Lambda) (1 - K(p/\Lambda)), \\ R_\Lambda(p) = p^2 \frac{K(p/\Lambda)}{1 - K(p/\Lambda)}, \end{cases} \quad (\text{A12})$$

where  $K(p)$  satisfies

$$K(p) \longrightarrow \begin{cases} 1 & (p \rightarrow 0), \\ 0 & (p \rightarrow \infty). \end{cases} \quad (\text{A13})$$

(Only the massless case is considered here for simplicity.) Regarding

$$S_{F\Lambda}[\phi] = -\frac{1}{2} \int_p \frac{p^2}{K(p/\Lambda)} \phi(p) \phi(-p) \quad (\text{A14})$$

as the free part of the Wilson action, we can develop perturbation theory.

With the above cutoff functions, (36) and (49) become

$$\begin{aligned}
-\Lambda \frac{\partial S_\Lambda}{\partial \Lambda} &= \int_p \left( \frac{\Delta(p/\Lambda)}{K(p/\Lambda)} - \gamma_\Lambda \right) \phi \frac{\delta S_\Lambda}{\delta \phi} \\
&\quad + \int_p \frac{1}{p^2} (\Delta(p/\Lambda) - 2\gamma_\Lambda K(1-K)) \frac{1}{2} \left\{ \frac{\delta S_\Lambda}{\delta \phi} \frac{\delta S_\Lambda}{\delta \phi} + \frac{\delta^2 S_\Lambda}{\delta \phi \delta \phi} \right\}, \quad (\text{A15})
\end{aligned}$$

$$-\Lambda \frac{\partial \Gamma_\Lambda}{\partial \Lambda} = -\gamma_\Lambda \int_p \Phi \frac{\delta \Gamma_\Lambda}{\delta \Phi} + \frac{1}{2} \int_p \frac{p^2}{(1-K)^2} (\Delta - 2\gamma_\Lambda K(1-K)) G_{\Lambda;p,-p}, \quad (\text{A16})$$

where  $\Delta(p/\Lambda) = \Lambda \partial_\Lambda K(p/\Lambda)$ , and (51) becomes

$$\int_p G_{\Lambda;p,-q} \left( q^2 \frac{K(q/\Lambda)}{1 - K(q/\Lambda)} \delta(q-r) - \frac{\delta^2 \Gamma_\Lambda}{\delta \Phi(q) \delta \Phi(-r)} \right) = \delta(p-r). \quad (\text{A17})$$

(A15) was derived in [8].

To make  $S_\Lambda$  equivalent to the action satisfying the Wilson convention, we must choose

$$K(p/\Lambda) = \frac{1}{1 + \frac{p^2}{\Lambda^2} \exp\left(\frac{2p^2}{\Lambda^2}\right)} \quad (\text{A18})$$

so that  $R_\Lambda(p) = \Lambda^2 \exp(-2p^2/\Lambda^2)$ . We then find

$$S_\Lambda^W[\phi] \equiv S_\Lambda \left[ \frac{e^{\frac{p^2}{\Lambda^2}}}{1 + \frac{p^2}{\Lambda^2} e^{\frac{2p^2}{\Lambda^2}}} \phi \right] \quad (\text{A19})$$

satisfies (A9). [19]

## B Comments on the results of Bervillier

In refs. [16–18] Bervillier gives ERG differential equations (for both Wilson and 1PI actions) with an anomalous dimension. He has only one arbitrary cutoff function, since he makes a choice

$$\begin{cases} K_\Lambda(p) = \tilde{P}(p/\Lambda) = K(p/\Lambda) \\ k_\Lambda(p) = \frac{p^2}{\Lambda^2} \tilde{P}(p/\Lambda) = \frac{p^2}{\Lambda^2} K(p/\Lambda) \end{cases} \quad (\text{B1})$$

so that

$$R_\Lambda(p) = \Lambda^2 \tilde{P}(p/\Lambda) = \Lambda^2 K(p/\Lambda). \quad (\text{B2})$$

Using the dimensionless notation, we can write Bervillier's ERG differential equations as

$$\begin{aligned} \partial_t S_t[\phi] &= \int_p \left\{ -p_\mu \frac{\partial \ln K(p)}{\partial p_\mu} + \frac{D+2}{2} - \gamma_t + p_\mu \frac{\partial}{\partial p_\mu} \right\} \phi(p) \cdot \frac{\delta S_t[\phi]}{\delta \phi(p)} \\ &+ \int_p \left\{ (2 - 2\gamma_t) K(p) - p_\mu \frac{\partial K(p)}{\partial p_\mu} \right\} \frac{1}{2} \left\{ \frac{\delta S_t[\phi]}{\delta \phi(p)} \frac{\delta S_t[\phi]}{\delta \phi(-p)} + \frac{\delta^2 S_t[\phi]}{\delta \phi(p) \delta \phi(-p)} \right\}, \end{aligned} \quad (\text{B3})$$

and

$$\begin{aligned} \partial_t \Gamma_t[\Phi] &= \int_p \left( \frac{D+2}{2} - \gamma_t + p_\mu \frac{\partial}{\partial p_\mu} \right) \Phi(p) \cdot \frac{\delta \Gamma_t[\Phi]}{\delta \Phi(p)} \\ &+ \int_p \left( (2 - 2\gamma_t) K(p) - p_\mu \frac{\partial K(p)}{\partial p_\mu} \right) \frac{1}{2} G_{t;p,-p}[\Phi], \end{aligned} \quad (\text{B4})$$

where

$$\int_q G_{t;p,-q}[\Phi] \left\{ K(q) \delta(q-r) - \frac{\delta^2 \Gamma_t[\Phi]}{\delta \Phi(q) \delta \Phi(-r)} \right\} = \delta(p-r). \quad (\text{B5})$$

(B2) simplifies considerably the Legendre transformation from  $S_t$  to  $\Gamma_t$ :

$$-\frac{1}{2} \int_p K(p) \Phi(p) \Phi(-p) + \Gamma_t[\Phi] = \frac{1}{2} \int_p \frac{1}{K(p)} \phi(p) \phi(-p) + S_t[\phi] - \int_p \phi(-p) \Phi(p). \quad (\text{B6})$$

To make Bervillier's Wilson action equivalent to Wilson's, we must choose

$$K(p/\Lambda) = \exp\left(-\frac{2p^2}{\Lambda^2}\right) \quad (\text{B7})$$

so that  $R_\Lambda(p) = \Lambda^2 \exp(-2p^2/\Lambda^2)$ . We then find

$$S_\Lambda^W[\phi] = S_\Lambda \left[ \frac{K(p/\Lambda)}{\exp(-p^2/\Lambda^2)} \phi \right] = S_\Lambda [\exp(-p^2/\Lambda^2) \phi] \quad (\text{B8})$$

satisfies (A9), as is explained in [16].

## C Comments on the results of Rosten and those of Osborn and Twigg

Osborn and Twigg [13] and independently Rosten [14] have considered the ERG differential equation of Ball et al. [15] given by

$$\begin{aligned} -\Lambda \frac{\partial}{\partial \Lambda} S_\Lambda[\phi] &= \int_p \left( \frac{\Delta(p/\Lambda)}{K(p/\Lambda)} - \gamma_\Lambda \right) \phi(p) \frac{\delta S_\Lambda}{\delta \phi(p)} \\ &+ \int_p \frac{\Delta(p/\Lambda)}{p^2} \frac{1}{2} \left\{ \frac{\delta S_\Lambda}{\delta \phi(p)} \frac{\delta S_\Lambda}{\delta \phi(-p)} + \frac{\delta^2 S_\Lambda}{\delta \phi(p) \delta \phi(-p)} \right\}, \end{aligned} \quad (\text{C1})$$

where the dimensionful notation is used, and

$$\Delta(p/\Lambda) \equiv \Lambda \frac{\partial}{\partial \Lambda} K(p/\Lambda). \quad (\text{C2})$$

They succeeded in constructing a Legendre transformation that gives a 1PI action satisfying the ERG differential equation of the type (49). We wish to reproduce their result using our line of reasoning.

Comparing (C1) with (A15), we notice that (C1) is simpler since it is missing second order derivative terms proportional to  $\gamma_\Lambda$ . But the simplicity of (C1) is misleading; if we try to obtain the two cutoff functions satisfying

$$\begin{cases} \Lambda \frac{\partial \ln \sqrt{Z_\Lambda} K_\Lambda(p)}{\partial \Lambda} &= \frac{\Delta(p/\Lambda)}{K(p/\Lambda)} - \gamma_\Lambda, \\ \Lambda \frac{\partial}{\partial \Lambda} \ln \frac{Z_\Lambda K_\Lambda(p)^2}{k_\Lambda(p)} \cdot k_\Lambda(p) &= \Delta(p/\Lambda), \end{cases} \quad (\text{C3})$$

so that (C1) coincides with (36), we find the following rather complicated solutions (obtained in [14] and [13]):

$$\begin{cases} K_\Lambda(p) &= K(p/\Lambda) b(p), \\ k_\Lambda(p) &= Z_\Lambda K(p/\Lambda)^2 \left( \int_\Lambda^\mu \frac{d\Lambda'}{\Lambda'} \frac{1}{Z_{\Lambda'}} \frac{\Delta(p/\Lambda')}{K(p/\Lambda')^2} + a(p) \right), \end{cases} \quad (\text{C4})$$

where  $a(p), b(p)$  are arbitrary dimensionless functions of  $p^2$ , independent of  $\Lambda$ . ( $a(p)$  must be of order  $p^2/\Lambda^2$  near zero, and  $b(p)$  is positive. The  $\mu$  dependence of the integral can be

absorbed by  $a(p)$ .) Hence, we obtain

$$R_\Lambda(p) = p^2 \frac{b(p)^2}{Z_\Lambda \left( \int_\Lambda^{\Lambda_0} \frac{d\Lambda'}{\Lambda'} \frac{1}{Z_{\Lambda'}} \frac{\Delta(p/\Lambda')}{K(p/\Lambda')^2} + a(p) \right)}. \quad (\text{C5})$$

The Legendre transformation (32) gives the 1PI action as

$$\Gamma_\Lambda[\Phi] = S_\Lambda[\phi] + \frac{1}{2} \int_p R_\Lambda(p) \Phi(p) \Phi(-p) + \frac{1}{2} \int_p \frac{p^2}{k_\Lambda(p)} \phi(p) \phi(-p) - \int_p \frac{R_\Lambda(p)}{K_\Lambda(p)} \phi(-p) \Phi(p), \quad (\text{C6})$$

which satisfies (49) and (51) with the anomalous dimension.

For easier comparisons with the results of [13, 14], let us use the notation of [14] to express the above cutoff functions. We define

$$\sigma_\Lambda(p) \equiv Z_\Lambda K(p/\Lambda) \left( \int_\Lambda^\mu \frac{d\Lambda'}{\Lambda'} \frac{1}{Z_{\Lambda'}} \frac{\Delta(p/\Lambda')}{K(p/\Lambda')^2} + a(p) \right). \quad (\text{C7})$$

We then obtain

$$\mathcal{P}_\Lambda(p) \equiv \frac{R_\Lambda(p)}{K_\Lambda(p)} = p^2 \frac{b(p)}{\sigma_\Lambda(p)}, \quad (\text{C8})$$

$$\mathcal{Q}_\Lambda(p) \equiv p^2 \left( \frac{1}{k_\Lambda(p)} - \frac{1}{K(p/\Lambda)} \right) = \frac{p^2}{K(p/\Lambda)} \left( \frac{1}{\sigma_\Lambda(p)} - 1 \right), \quad (\text{C9})$$

$$\mathcal{R}_\Lambda(p) \equiv R_\Lambda(p) = p^2 \frac{K(p/\Lambda)}{\sigma_\Lambda(p)} b(p)^2, \quad (\text{C10})$$

corresponding to Rosten's  $c(p) = 0$ . Hence, denoting the interaction part by

$$S_{I\Lambda}[\phi] = S_\Lambda[\phi] + \frac{1}{2} \int_p \frac{p^2}{K(p/\Lambda)} \phi(p) \phi(-p), \quad (\text{C11})$$

we can rewrite the above Legendre transformation as

$$\begin{aligned} \Gamma_\Lambda[\Phi] &= S_{I\Lambda}[\phi] \\ &+ \int_p \left( -\mathcal{P}_\Lambda(p) \phi(-p) \Phi(p) + \frac{1}{2} \mathcal{Q}_\Lambda(p) \phi(p) \phi(-p) + \frac{1}{2} \mathcal{R}_\Lambda(p) \Phi(p) \Phi(-p) \right), \end{aligned} \quad (\text{C12})$$

agreeing with (17) of [14].

The dimensionless ERG differential equation for the 1PI action is obtained as

$$\begin{aligned}
0 &= \int_p \left( \frac{D+2}{2} - \gamma^* + p_\mu \frac{\partial}{\partial p_\mu} \right) \Phi(p) \cdot \frac{\delta \Gamma^*[\Phi]}{\delta \Phi(p)} \\
&\quad + \int_p \left( -p_\mu \frac{\partial \ln R^*(p)}{\partial p_\mu} + 2 - 2\gamma^* \right) R^*(p) \frac{1}{2} G_{p,-p}^*[\Phi]
\end{aligned} \tag{C13}$$

and

$$\int_q G_{p,-q}^*[\Phi] \left\{ R^*(q) \delta(q-r) - \frac{\delta^2 \Gamma^*[\Phi]}{\delta \Phi(q) \delta \Phi(-r)} \right\} = \delta(p-r), \tag{C14}$$

where

$$\frac{1}{R^*(p)} \equiv \frac{1}{b(0)^2} \frac{1}{p^2} \int_0^\infty dt e^{2\gamma t} \frac{\Delta(pe^{-t})}{K(pe^{-t})^2}. \tag{C15}$$

## D Perturbative examples

We would like to sketch two perturbative calculations of the anomalous dimensions using ERG. We will use the Polchinski convention [21] of Sect. A.2.2:

$$K_\Lambda(p) = K\left(\frac{p}{\Lambda}\right), \quad k_\Lambda(p) = K\left(\frac{p}{\Lambda}\right) \left(1 - K\left(\frac{p}{\Lambda}\right)\right). \tag{D1}$$

We do not need to specify an explicit form of  $K$  for the 1-loop calculations. Denoting

$$\Delta\left(\frac{p}{\Lambda}\right) \equiv \Lambda \frac{\partial}{\partial \Lambda} K\left(\frac{p}{\Lambda}\right), \tag{D2}$$

we obtain

$$\int_p \frac{1}{p^D} \Delta\left(\frac{p}{\Lambda}\right) \left(1 - K\left(\frac{p}{\Lambda}\right)\right)^n = \frac{\Omega_{D-1}}{(2\pi)^D} \int_0^\infty dp \frac{1}{n+1} \frac{d}{dp} (1 - K(p))^{n+1} = \frac{\Omega_{D-1}}{(2\pi)^D} \frac{1}{n+1}, \tag{D3}$$

where  $\Omega_{D-1}$  is the volume of the unit  $D-1$  sphere given by

$$\Omega_{D-1} \equiv \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)}. \tag{D4}$$

### D.1 $\phi^3$ theory in $D=6$

Let us consider a real scalar theory in  $D=6$  whose classical action is given by

$$S_{cl} = - \int d^6x \left( \frac{1}{2} (\partial_\mu \phi)^2 + \frac{g}{3!} \phi^3 \right). \tag{D5}$$

Let us denote the two-point vertex function of  $S_\Lambda$  by

$$\int_p A_2(p) \frac{1}{2} \phi(p) \phi(-p). \tag{D6}$$

At 1-loop, (36) gives

$$-\Lambda \frac{\partial A_2(p)}{\partial \Lambda} = 2\gamma_\Lambda p^2 + \frac{g^2}{2} \int_q \frac{\Delta(q/\Lambda)}{q^2} \frac{1 - K\left(\frac{p+q}{\Lambda}\right)}{(p+q)^2}. \quad (\text{D7})$$

We determine  $\gamma_\Lambda$  so that the coefficient of  $p^2$  vanishes. Expanding the above to order  $p^2$ , we obtain

$$\gamma_\Lambda = \frac{g^2}{4} \int_q \frac{\Delta(q)}{q^6} \frac{2}{3} (1 - K(q)) = \frac{g^2}{(4\pi)^3} \frac{1}{12}. \quad (\text{D8})$$

## D.2 Yukawa theory in $D = 4$

We next consider a theory in  $D = 4$  with a real scalar and a Dirac field. The classical action is given by

$$S_{cl} = - \int d^4x \left( \bar{\psi} \frac{1}{i} \not{\partial} \psi + \frac{1}{2} (\partial_\mu \phi)^2 + ig \phi \bar{\psi} \psi \right). \quad (\text{D9})$$

The generalization of the ERG differential equation (36) for this case is obtained as

$$\begin{aligned} -\Lambda \frac{\partial S_\Lambda[\phi, \psi, \bar{\psi}]}{\partial \Lambda} &= \int_p \left( \Lambda \frac{\partial \ln K_\Lambda(p)}{\partial \Lambda} - \gamma_B \right) \phi(p) \frac{\delta S_\Lambda}{\delta \phi(p)} \\ &+ \int_p \left( \Lambda \frac{\partial \ln K_\Lambda(p)}{\partial \Lambda} - \gamma_F \right) \left( S_\Lambda \frac{\overleftarrow{\delta}}{\delta \psi(p)} \psi(p) + \bar{\psi}(-p) \frac{\overrightarrow{\delta}}{\delta \bar{\psi}(-p)} S_\Lambda \right) \\ &+ \int_p \left( \Lambda \frac{\partial \ln R_\Lambda(p)}{\partial \Lambda} - 2\gamma_B \right) \frac{k_\Lambda(p)}{p^2} \frac{1}{2} \left\{ \frac{\delta S_\Lambda}{\delta \phi(p)} \frac{\delta S_\Lambda}{\delta \phi(-p)} + \frac{\delta^2 S_\Lambda}{\delta \phi(p) \delta \phi(-p)} \right\} \\ &+ \int_p \left( \Lambda \frac{\partial \ln R_\Lambda(p)}{\partial \Lambda} - 2\gamma_F \right) k_\Lambda(p) \left\{ S_\Lambda \frac{\overleftarrow{\delta}}{\delta \psi(p)} \frac{1}{p} \frac{\overrightarrow{\delta}}{\delta \bar{\psi}(-p)} S_\Lambda - \text{Tr} \frac{1}{p} \frac{\overrightarrow{\delta}}{\delta \bar{\psi}(-p)} S_\Lambda \frac{\overleftarrow{\delta}}{\delta \psi(p)} \right\}, \end{aligned} \quad (\text{D10})$$

where  $\gamma_B, \gamma_F$  are the  $\Lambda$ -dependent anomalous dimensions of the scalar and Dirac fields, respectively. For simplicity, we have used the same cutoff functions for the scalar and Dirac fields.

Let us denote the two-point vertex functions of  $S_\Lambda$  by

$$\int_p \left( A_B(p) \frac{1}{2} \phi(p) \phi(-p) + \bar{\psi}(-p) A_F(p) \psi(p) \right). \quad (\text{D11})$$

At 1-loop, we obtain

$$\begin{aligned}
-\Lambda \frac{\partial}{\partial \Lambda} A_B(p) &= 2\gamma_B p^2 + g^2 \int_q \text{Tr} \frac{1}{\not{q}} \frac{1}{\not{q} + \not{p}} \left\{ \Delta \left( \frac{p+q}{\Lambda} \right) \left( 1 - K \left( \frac{q}{\Lambda} \right) \right) \right. \\
&\quad \left. + \left( 1 - K \left( \frac{p+q}{\Lambda} \right) \right) \Delta \left( \frac{q}{\Lambda} \right) \right\}, \tag{D12}
\end{aligned}$$

$$\begin{aligned}
-\Lambda \frac{\partial}{\partial \Lambda} A_F(p) &= 2\gamma_F i \not{p} - g^2 \int_q \frac{1}{q^2 (\not{q} + \not{p})} \left\{ \Delta \left( \frac{p+q}{\Lambda} \right) \left( 1 - K \left( \frac{q}{\Lambda} \right) \right) \right. \\
&\quad \left. + \left( 1 - K \left( \frac{p+q}{\Lambda} \right) \right) \Delta \left( \frac{q}{\Lambda} \right) \right\}. \tag{D13}
\end{aligned}$$

We choose  $\gamma_{B,F}$  to cancel the kinetic terms. We then obtain

$$\gamma_B = 2g^2 \int_q \frac{\Delta(q) (1 - K(q))}{q^4} = 2 \frac{g^2}{(4\pi)^2}, \tag{D14}$$

$$\gamma_F = \frac{g^2}{2} \int_q \frac{\Delta(q) (1 - K(q))}{q^4} = \frac{1}{2} \frac{g^2}{(4\pi)^2}. \tag{D15}$$